Further Evidence for Xie's Inequality

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Abstract

If the inequality (with constant already proven to be optimal)

$$\sup_{\Omega} |u|^2 \leq \frac{1}{3\pi} \|\nabla u\| \left\| \widetilde{\triangle} u \right\| \tag{1}$$

is valid for every open set $\Omega \subset R^3$, without any conditions on the regularity of the boundary, then so is most of the current existence and regularity theory for the Navier-Stokes equations. Here u is assumed to belong to the completion $J_0(\Omega)$ of $D(\Omega) = \{\varphi \in C_0^\infty(\Omega) : \nabla \cdot \varphi = 0\}$ in the Dirichlet norm $\|\nabla \cdot\|$, and to have a square-integrable Stokes operator $\widetilde{\triangle} u$.

In proving an analogue of (1) for the Laplacian, Xie used the maximum principle to show that for every point y in a smoothly bounded domain Ω,

$$\|G_\mu(\cdot, y)\|_{L^2(\Omega)} \leq \|g_\mu(\cdot, y)\|_{L^2(R^3)}, \tag{2}$$

where G_μ is the Green's function for the Helmholtz operator $-\triangle + \mu$, and g_μ is the corresponding fundamental singularity. Although he conjectured a similar inequality for the Stokes operator, it has yet to be proven. However, we have proven, even for the Stokes operator, that the ratio of the two sides of (2) tends to 1 as $\mu \to \infty$. Consequently, to complete the proof of (1), it is now enough to consider smoothly bounded domains and to show a tendency toward singularity,

$$\mu_n \equiv \left\| \widetilde{\triangle} u_n \right\|^2 / \|\nabla u_n\|^2 \to \infty, \tag{3}$$

if $\{u_n\}$ is a sequence of functions such that the ratio of the left to right sides of (1) tends to its supremum. Our purpose here is to give two new results providing suggestive evidence for this.

Using the argument Xie introduced in proving his analogue of (1) for the Laplacian, we prove a sharp inequality for the sup-norm of a function $f(x) = \sum_{n=1}^{\infty} c_n \cos nx$, in terms of the L^2-norms of its fractional derivatives of orders 1/3 and 2/3, namely

$$\sup_{(-\pi,\pi)} |f|^2 \leq 3 \left\| f^{(1/3)} \right\|_{L^2(-\pi,\pi)} \left\| f^{(2/3)} \right\|_{L^2(-\pi,\pi)}. \tag{4}$$

In proving (4), we consider a sequence $\{f_n\}$ of functions such that the ratio of the left to right sides of (4) tends to its supremum. For this sequence the analogue of (3) holds, that is,

$$\mu_n \equiv \left\| f_n^{(2/3)} \right\|_{L^2(-\pi,\pi)}^2 / \left\| f_n^{(1/3)} \right\|_{L^2(-\pi,\pi)}^2 \to \infty. \tag{5}$$

Our proof of (5) is based on a demonstration that

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{1}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1 \right] < 0, \qquad \text{for all } \mu \in [0, \infty). \tag{6}$$

This is an analogue of conjectures about the eigenfunctions of the Stokes operator and the Laplacian that we have previously shown to imply (3) and its analogue for the Laplacian. For the Laplacian, the conjecture is that for any fixed point $y \in \Omega$,

$$f_\infty(\mu, y) \equiv \sum_{n=1}^{\infty} \frac{\phi_n^2(y)}{\left(\mu + \lambda_n\right)^2} \left[\frac{\mu}{\lambda_n} - 1 \right] < 0, \qquad \text{for all } \mu \in [0, \infty), \tag{7}$$

where the $\{\phi_n\}$ are the L^2-orthonormal eigenfunctions of $\triangle$, satisfying $-\triangle \phi_n = \lambda_n \phi_n$, $\phi_n|_{\partial\Omega} = 0$. In this report we prove (7) for points y such that, for some R that may depend on y, $\{x : |x - y| < R/2\} \subset \Omega \subset \{x : |x - y| < R\}$. We suspect, of course, that the tendency toward singularity in a maximizing sequence is generic, and extends to the Stokes operator, implying (3) and thus (1).